C Time independent integral

In this appendix we show that when $T_b{}^* = G(u/t)$, $T_b(t)$ must be proportional to t^{μ} in which case the integral I [eq.(2.53)] is time independent. Subsequently, I is calculated as a function of μ.

We start with a general function $f(x)$ defined for $x > 0$. The assumption is that

$$\frac{f(x)}{f(y)} = g\left(\frac{x}{y}\right). \tag{C.1}$$

This implies that

$$\frac{f(x)}{f(1)} = g(x) = \frac{f(xy)}{f(y)}. \tag{C.2}$$

Then

$$f(xy) = g(x) \cdot f(y) = \frac{f(x)}{f(1)} \cdot f(y). \tag{C.3}$$

When

$$h(x) \equiv \frac{f(x)}{f(1)}, \tag{C.4}$$

then

$$h(xy) = \frac{f(xy)}{f(1)} = \frac{f(x) \cdot f(y)}{f(1) \cdot f(1)} = h(x) \cdot h(y). \tag{C.5}$$

When h is a continuous function then

$$h(x^{\alpha}) = (h(x))^{\alpha}, \tag{C.6}$$

and in particular

$$h(e^{\alpha}) = (h(e))^{\alpha}. \tag{C.7}$$

For $y = e^{\alpha}$, $\alpha = \ln y$ and, necessarily,

$$h(y) = (h(e))^{\ln y} = (e^{\ln h(e)})^{\ln y} = (e^{\ln y})^{\ln h(e)} = y^{\ln h(e)}. \tag{C.8}$$

It follows that

$$f(x) = f(1) \cdot h(x) = f(1) \cdot x^{\ln h(e)} = A \cdot x^{\mu}. \tag{C.9}$$

When $T_b(t) \propto t^{\mu}$, $T_b{}^* = (u/t)^{\mu}$ and I is easily calculated. In terms of the variable $y \equiv \sqrt{1 - u/t}$, for which eq.(2.53) becomes

$$I = \int_0^1 (1-y^2)^{\mu} dy, \tag{C.10}$$

we find for the value μ the relation

$$\begin{aligned} I_{\mu} &= \int_0^1 (1-y^2)^{\mu} dy = y(1-y^2)^{\mu}\Big|_0^1 - \int_0^1 -2\mu y^2 (1-y^2)^{\mu-1} dy \\ &= 2\mu \int_0^1 y^2 (1-y^2)^{\mu-1} dy = 2\mu \left\{ -\int_0^1 (1-y^2)^{\mu} dy + \int_0^1 (1-y^2)^{\mu-1} dy \right\} \\ &= 2\mu \{ I_{\mu-1} - I_{\mu} \}. \end{aligned} \tag{C.11}$$

Consequently

$$I_{\mu} = \frac{2\mu}{1+2\mu} I_{\mu-1}. \tag{C.12}$$

With $I_0 = 1$ and $I_{1/2} = \frac{\pi}{4}$, $I_{k/2}$ for k is a positive integer can be calculated easily.